

GREEN OPERATORS IN THE EDGE CALCULUS

B.-W. SCHULZE AND A.VOLPATO

ABSTRACT. The task to construct parametrices of elliptic differential operators on a manifold with edges requires a calculus of operators with a two-component principal symbolic hierarchy, consisting of (edge-degenerate) interior and (operator-valued) edge symbols. This so-called edge-algebra can be interpreted as a generalisation of the (pseudo-differential) algebra of boundary value problems without the transmission property at the boundary. We study new properties of the edge-algebra, in particular, what concerns the role of Green operators and their kernel representations.

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INTRODUCTION

This paper is aimed at characterising the structure of Green operators that occur in Green's functions of elliptic boundary value problems (without or with the transmission property at the boundary) or, more generally, in elliptic (so-called) edge problems. A boundary or an edge represents a singularity of the configuration,

and this causes a typical singular behaviour of solutions to such problems, here expressed in terms of discrete or continuous asymptotics. The Green operators just encode such asymptotics. At the same time it is interesting to understand the kernel structure of trace and potential operators in the boundary (edge) calculus; this is of a similar kind as the one of Green operators.

By a manifold M with edge Y we understand a topological space such that $M \setminus Y$ and Y are C^∞ manifolds, and Y has a neighbourhood V in M equipped with the structure of an X^Δ -bundle over Y for a closed compact C^∞ manifold X ; here $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ is the infinite cone with base X . In addition V is the quotient space of an $\overline{\mathbb{R}}_+ \times X$ -bundle \mathbb{V} over Y where $\mathbb{V} \rightarrow V$ is defined by the fibrewise projection $\overline{\mathbb{R}}_+ \times X \rightarrow X^\Delta$. From the definition it follows that M itself is the quotient space of a C^∞ manifold \mathbb{M} with boundary $\partial\mathbb{M}$, and $\partial\mathbb{M}$ is an X -bundle over Y . Locally near Y the manifold M has the structure of a wedge $X^\Delta \times \Omega$, $\Omega \subseteq \mathbb{R}^q$ open, $q = \dim Y$, and \mathbb{M} , the so-called stretched manifold of M , is locally near $\partial\mathbb{M}$ of the form $\overline{\mathbb{R}}_+ \times X \times \Omega$. The starting point of our discussion are differential operators on $\mathbb{M}_{\text{reg}} := \mathbb{M} \setminus \partial\mathbb{M}$ with smooth coefficients that are edge-degenerate near $\partial\mathbb{M}$, i.e., locally, in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ of the form

$$(0.0.1) \quad A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) \left(-r \frac{\partial}{\partial r} \right)^j (r D_y)^\alpha$$

with coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \text{Diff}^{\mu-(j+|\alpha|)}(X))$. Here $\text{Diff}^\nu(\cdot)$ denotes the space of all differential operators of order ν on the manifold in parentheses. If A is elliptic in the sense of standard ellipticity in \mathbb{M}_{reg} , together with the condition that for every fixed y the expression

$$\sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y) (-i\rho)^j \eta^\alpha$$

represents a parameter-dependent elliptic family of differential operators on X with the parameters $(\rho, \eta) \in \mathbb{R}^{1+q}$ (in the sense of Agranovich and Vishik, cf. [1]), we may ask the (pseudo-differential) nature of a parametrix of A and the Fredholm property in suitable weighted Sobolev spaces on \mathbb{M} . It turns out that such a characterisation of the solvability properties of the equation $Au = f$ induces a large variety of additional structures of an ‘unexpected’ complexity.

Comparing the situation with boundary value problems (taking into account that a, say, compact, C^∞ manifold M with boundary is a particular manifold with edge, where the above mentioned cone bundle near the edge ∂M can be identified with the inner normal bundle) we see that the solvability should be discussed in connection with extra conditions of trace (and, in general, also potential) type with respect to the edge, analogously as elliptic boundary conditions. In addition the edge-degenerate nature of the operators makes it necessary to refer to specific weighted edge spaces rather than ‘standard’ Sobolev spaces. This aspect requires some special attention, and there are many possible choices of such spaces. Answers of the above mentioned solvability problem (together with the construction of the additional edge conditions) for a certain special class of such spaces, including the

characterisation of the structure of parametrices, may be found in [12], [13].

Analogously as for the case with boundary, in the case of a manifold with edge we talk about an edge problem for A . To be more precise, by an edge problem we understand the construction (or characterisation) of a solution u to the equation $Au = f$ on a manifold with edge Y under suitable trace (or edge) conditions on u at Y and, if necessary, additional potentials Kv of distributions v on Y which are to be added to Au in the image of A .

Formally, Green operators have an analogous origin as Green's functions in the parametrices of elliptic boundary value problems (Green's function has the form of a sum $E + G$ where E is a fundamental solution or a parametrix of the given elliptic operator and G a Green operator in our sense). However, in the edge case (or when the transmission property is violated) the Green operators are 'loaded' with a huge variety of asymptotic information at the edge, coming from the zeros of the conormal symbol of the given operator, see the formula (0.0.3) below. In the special case of boundary value problems with the transmission property the asymptotics are nothing other than smoothness up to the boundary, see, for instance, [3], and kernel characterisations of corresponding Green symbols may be found in [14]. Kernel characterisations for Green operators of the cone calculus are given in [16]; the case of Green edge symbols is treated in [15], although this refers to projective tensor products of spaces with discrete asymptotics; here we take another (completely adequate but more convenient) tensor product and also study the case of continuous asymptotics, see also [18]. The discrete case makes sense for y -independent asymptotics. In the y -dependent case it is reasonable to formulate the phenomena in the frame of continuous asymptotics (details will be explained below). Green operators are smoothing on \mathbb{M}_{reg} . Close to $\mathbb{M}_{\text{sing}} := \partial\mathbb{M}$ they can be characterised (modulo a certain kind of global smoothing operators of the edge calculus) as pseudo-differential operators on the edge with (classical) operator-valued symbols, with 'twisted' homogeneity. Green operators formally appear in a pseudo-differential calculus containing the operator A together with its parametrix in the elliptic case. To illustrate that a little more we write

$$A = \text{Op}(a) = F_{\eta \rightarrow y}^{-1} a(y, \eta) F_{y \rightarrow \eta},$$

(also denoted by $\text{Op}_y(a)$) with the Fourier transform F in $y \in \mathbb{R}^q$, and

$$(0.0.2) \quad a(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) \left(-r \frac{\partial}{\partial r} \right)^j (r\eta)^\alpha.$$

Here, $a(y, \eta)$ is interpreted as an operator-valued amplitude function, operating in suitable weighted distribution spaces on the infinite (open stretched) cone $X^\wedge := \mathbb{R}_+ \times X$ with base X . Since the specific information is coming from a neighbourhood of $r = 0$, we may (and will) assume that the coefficients $a_{j\alpha}(r, y)$ are independent of r for $r > R$ for some $R > 0$. The behaviour of A near the edge Y very much depends on the so-called principal edge symbol

$$\sigma_\wedge(A)(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y) \left(-r \frac{\partial}{\partial r} \right)^j (r\eta)^\alpha,$$

$(y, \eta) \in T^*Y \setminus 0$, and the subordinate conormal symbol

$$(0.0.3) \quad \sigma_c \sigma_\wedge(A)(y, z) := \sum_{j=0}^{\mu} a_{j0}(0, y) z^j$$

which is (in the present case of a differential operator) an entire function in $z \in \mathbb{C}$ taking values in $\text{Diff}^\mu(X)$. As such it is a holomorphic family of Fredholm operators

$$(0.0.4) \quad \sigma_c \sigma_\wedge(A)(y, z) : H^s(X) \longrightarrow H^{s-\mu}(X)$$

in standard Sobolev spaces on X . Parametrix of A are to a large extent determined by the inverse of (0.0.4) which is a meromorphic operator function. Poles and multiplicities may depend on y ; this gives rise to clouds of points in the complex plane, and the y -dependent nature of meromorphy can be described in terms of (operator-valued) analytic functionals in the complex plane, pointwise discrete and branching, but smoothly depending on the edge-variable y (with respect to the Fréchet topology of the space of analytic functionals). Now the Green operators of the edge calculus which participate in the parametrices of elliptic operators inherit a part of this structure and affect the elliptic regularity of solutions. To briefly recall the idea, an equation $Au = f$ with elliptic A is multiplied from the left by a parametrix P of A ; then using $PA = I - G$ for a Green operator G we obtain

$$(0.0.5) \quad PAu = f \quad \Rightarrow \quad u = Pf + Gu.$$

If P is sensitive enough (which is the case in the corresponding edge calculus) to transform a given f with asymptotics to a function Pf with some resulting asymptotics and if G produces from any weighted distribution u (without asymptotics) a smooth function with asymptotics, from (0.0.5) we immediately conclude the asymptotics of u . Note that a similar philosophy works in every calculus with parametrices of elliptic elements (with or without asymptotics), for instance, in the edge calculus of [4], where the base spaces of model cones may have non-constant dimension.

In other words, it is interesting to understand the nature of Green operators in the edge calculus. Let us also note that the above-mentioned trace and potential operators with respect to the edge can also be subsumed under the concept of Green operators.

1. EDGE SYMBOLS AND WEIGHTED SOBOLEV SPACES

We study Mellin operators in the distance variable $r \in \mathbb{R}_+$ to the singularity and recall the edge quantisation of edge-degenerate symbols in r -direction. After that we formulate the concept of discrete and continuous asymptotics for $r \rightarrow 0$ in weighted cone Sobolev spaces. Green symbols of the edge calculus are formulated as classical symbols with twisted homogeneity, mapping weighted distributions into smooth functions with such asymptotics.

1.1. Operator functions on the infinite stretched cone. Let X be a closed compact C^∞ manifold, and let $L_{\text{cl}}^\mu(X; \mathbb{R}^l)$ denote the space of classical parameter-dependent pseudo-differential operators on X of order $\mu \in \mathbb{R}$, with the parameter $\lambda \in \mathbb{R}^l$ (that is, the local amplitude functions are classical symbols in $(\xi, \lambda) \in \mathbb{R}^{n+l}$, $n = \dim X$, and $L^{-\infty}(X; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(X))$, where $L^{-\infty}(X)$ is the space of smoothing operators on X , identified with the space of operators with kernels in

$C^\infty(X \times X)$ via a Riemannian metric on X). Analogously as the operator functions (0.0.2) in the pseudo-differential case we consider

$$r^{-\mu} \text{op}_r(p)(y, \eta)$$

for an operator family p of the form

$$p(r, y, \rho, \eta) := \tilde{p}(r, y, r\rho, r\eta)$$

where $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L_{\text{cl}}^\mu(X; \mathbb{R}_{\tilde{\rho}, \tilde{\eta}}^{1+q}))$ and

$$\text{op}_r(p)(y, \eta)u(r) = \iint e^{i(r-r')\rho} p(r, y, \rho, \eta)u(r')dr' d\rho.$$

Let M denote the Mellin transform on \mathbb{R}_+ , i.e., $Mu(z) := \int_0^\infty r^{z-1}u(r)dr$, first for $u \in C_0^\infty(\mathbb{R}_+)$ and then extended to more general function and distribution spaces, also vector-valued ones. Concerning details on the Mellin transform in connection with edge-degenerate operators, see [13] or [6]. Set $\Gamma_\delta := \{z \in \mathbb{C} : \text{Re } z = \delta\}$ for any $\delta \in \mathbb{R}$, and define the weighted Mellin transform with weight γ as the map $M_\gamma : u \mapsto Mu(z)|_{\Gamma_{\frac{1}{2}-\gamma}}$, $u \in C_0^\infty(\mathbb{R}_+)$. With M_γ we can associate weighted Mellin pseudo-differential operators $\text{op}_M^\gamma(f)$ with scalar as well as operator-valued amplitude functions f . Note that Mellin operators have been widely used in Eskin's monograph [7] in connection with boundary value problems without the transmission property, see also [9]. In the edge calculus we take parameter-dependent families of pseudo-differential operators $f(r, y, z, \eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L_{\text{cl}}^\mu(X; \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^q))$; here $(z, \eta) \in \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^q$ plays the role of parameters while (r, y) are additional variables. The associated operators have the form

$$(1.1.1) \quad \text{op}_M^\gamma(f)(y, \eta)u(r) := \iint \left(\frac{r}{r'}\right)^{-\left(\frac{1}{2}-\gamma+i\rho\right)} f(r, y, \frac{1}{2}-\gamma+i\rho, \eta)u(r')\frac{dr'}{r'}d\rho.$$

In the example of (0.0.2) we can write

$$\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) = \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)(-i\tilde{\rho})^j \tilde{\eta}^\alpha;$$

then $a(y, \eta) = r^{-\mu} \text{op}_r(p)(y, \eta)$, and

$$a(y, \eta) := r^{-\mu} \text{op}_M^\beta(f)(y, \eta)$$

for $f(r, y, z, \eta) := \tilde{f}(r, y, z, r\eta)$, $\tilde{f}(r, y, z, \tilde{\eta}) = \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)z^j \tilde{\eta}^\alpha$ and any $\beta \in \mathbb{R}$ (at this moment $a(y, \eta)$ is interpreted as an operator function in the sense $C_0^\infty(X^\wedge) \longrightarrow C^\infty(X^\wedge)$; later on we deal with weighted spaces on $X^\wedge = \mathbb{R}_+ \times X$).

Observe that the correspondence $p \rightarrow f$ in the case of a differential operator (0.0.1) is canonical, i.e., we have in this case

$$A = r^{-\mu} \text{Op}_y(\text{op}_r(p)) = r^{-\mu} \text{Op}_y(\text{op}_M^\gamma(f)),$$

for any real γ , or, alternatively,

$$(1.1.2) \quad \text{op}_r(p)(y, \eta) = \text{op}_M^\gamma(f)(y, \eta)$$

as elements of $C^\infty(\Omega, L_{\text{cl}}^\mu(X^\wedge; \mathbb{R}^q))$. However, in the pseudo-differential case the correspondence $p \rightarrow f$ has to be achieved by a more subtle quantisation which leaves smoothing remainders that are not necessarily vanishing. It is known that such quantisations exist, indeed, see [13], and that for every p there are elements $\tilde{f}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L_{\text{cl}}^\mu(X; \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^q))$ such that for the associated f the

relation (1.1.2) holds modulo such remainders. We even find holomorphic functions in z , see also Section 3.2, and (1.1.2) then holds for arbitrary $\gamma \in \mathbb{R}$. The pseudo-differential edge calculus of [13] contains amplitude functions of the form

$$\begin{aligned} a(y, \eta) &:= r^{-\mu} \sigma \{ \omega(r[\eta]) \text{op}_M^{\gamma - \frac{n}{2}}(f)(y, \eta) \tilde{\omega}(r[\eta]) \\ &\quad + (1 - \omega(r[\eta])) \text{op}_r(p)(y, \eta) (1 - \tilde{\omega}(r[\eta])) \} \tilde{\sigma}, \end{aligned}$$

$n = \dim X$, where $\omega, \tilde{\omega}, \tilde{\tilde{\omega}}$ and $\sigma, \tilde{\sigma}$ are cut-off functions on the half axis, i.e., elements in $C_0^\infty(\overline{\mathbb{R}}_+)$ which are equal to one near zero, and with the property $\tilde{\tilde{\omega}} \prec \omega \prec \tilde{\omega}$, where $\varphi \prec \psi$ means that $\psi \equiv 1$ in a neighbourhood of $\text{supp } \varphi$. In addition $\eta \rightarrow [\eta]$ is a strictly positive C^∞ function in $\eta \in \mathbb{R}^q$ such that $[\eta] = |\eta|$ for $|\eta| > C$ for some $C > 0$.

In order to complete operators $\text{Op}_y(a)$ to an algebra we have to add other operator-valued symbols, mainly (so-called) Green symbols and also smoothing Mellin operator-families, see Section 3.2 below. Those are encoding the asymptotic properties of solutions to elliptic equations when they are coming from the parametrix.

1.2. Weighted spaces with discrete asymptotics on cones. Green operators in the context of classical boundary value problems are (locally in a collar neighbourhood of the boundary) pseudo-differential operators along the boundary with symbols acting as operators normal to the boundary. More precisely, the values of the symbols are operators G in $L^2(\mathbb{R}_+)$ such that $G, G^* : L^2(\mathbb{R}_+) \rightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$ ($= \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$) are continuous (this concerns the so called type zero; otherwise the operators are combined with differentiations transversal to the boundary). In the generalisation to the case of a manifold with edges we replace the inner normal $\overline{\mathbb{R}}_+$ by a non-trivial model cone $X^\Delta := (\overline{\mathbb{R}}_+ \times X) / (\{0\} \times X)$ belonging to corresponding local wedges, and the spaces $L^2(\mathbb{R}_+)$ and $\mathcal{S}(\overline{\mathbb{R}}_+)$ by weighted spaces $\mathcal{K}^{0,\gamma}(X^\Delta)$ and $\mathcal{S}_{\mathcal{P}}^\beta(X^\Delta)$, respectively, on the open stretched cone $X^\Delta := \mathbb{R}_+ \times X \ni (r, x)$ with a certain behaviour for $r \rightarrow 0$, encoded by a so called asymptotic type \mathcal{P} . In simplest cases asymptotics will have the form

$$(1.2.1) \quad u(r, x) \sim \sum_j \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r \quad \text{for } r \rightarrow 0$$

with a sequence of triples $\mathcal{P} := \{(p_j, m_j, L_j)\}_{j=0,1,\dots,N}$, $N \in \mathbb{N} \cup \{\infty\}$, $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and finite-dimensional subspaces $L_j \subset C^\infty(X)$, such that $c_{jk} \in L_j$ for all $0 \leq k \leq m_j$, and all j . For $\dim X = 0$ we have, in particular, a natural identification of $\mathcal{S}(\overline{\mathbb{R}}_+)$ with $\mathcal{S}_T^0(\mathbb{R}_+)$ for the Taylor asymptotic type $T = \{(-j, 0)\}_{j=0,1,\dots}$ (the spaces L_j disappear in this case). Let us now pass to the precise definitions.

We say that \mathcal{P} is associated with weight data (γ, Θ) for a weight $\gamma \in \mathbb{R}$ and $\Theta = (\vartheta, 0]$ for some $-\infty \leq \vartheta < 0$ if the set $\pi_{\mathbb{C}} \mathcal{P} = \{p_j\}_{0 \leq j \leq N}$ is contained in the strip $\{\frac{n+1}{2} - \gamma + \vartheta < \text{Re } z < \frac{n+1}{2} - \gamma\}$, $n = \dim X$, $\pi_{\mathbb{C}} \mathcal{P}$ finite for finite ϑ , and $\text{Re } p_j \rightarrow -\infty$ as $j \rightarrow \infty$ when $\vartheta = -\infty$ and $N = \infty$.

Given \mathcal{P} associated with (γ, Θ) for a finite weight interval Θ we set

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) := \left\{ \sum_{j=0}^N \sum_{k=0}^{m_j} \omega(r) c_{jk} r^{-p_j} \log^k r : c_{jk} \in L_j \text{ for } 0 \leq k \leq m_j, 0 \leq j \leq N \right\}.$$

Moreover, let $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, denote the subspace of all $u(r, x) \in r^{\gamma - \frac{n}{2}} L^2(X^\wedge)$ (with L^2 referring to $dr dx$) such that

$$(r \partial_r)^k D_x^\alpha u(r, x) \in r^{\gamma - \frac{n}{2}} L^2(X^\wedge)$$

for every $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, $k + |\alpha| \leq s$; here $D_x^\alpha := v_1^{\alpha_1} \dots v_n^{\alpha_n}$ means the differentiation with arbitrary vector fields v_j on X . In particular, we have $\mathcal{H}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(X^\wedge)$. Then duality and interpolation give us a definition of $\mathcal{H}^{s,\gamma}(X^\wedge)$ for arbitrary $s \in \mathbb{R}$.

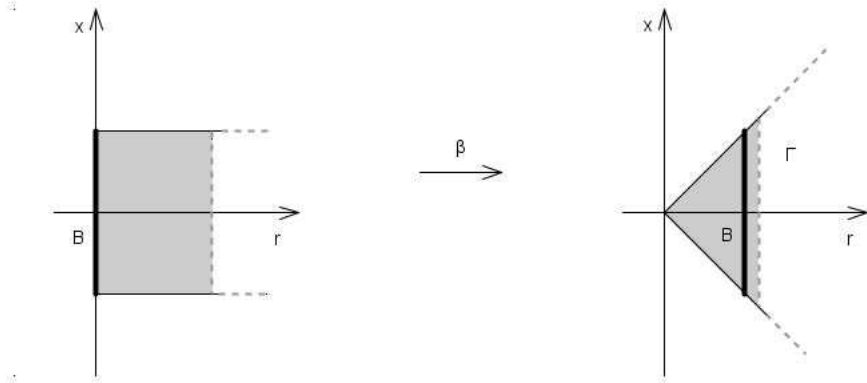
There is another useful scale of weighted spaces on X^\wedge defined by

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega f + (1 - \omega)g : f \in \mathcal{H}^{s,\gamma}(X^\wedge), g \in H_{\text{cone}}^s(X^\wedge)\}$$

for some cut-off function ω ; see, for instance, [10] or [13]. The space $H_{\text{cone}}^s(X^\wedge)$ is defined to be the set of all $g \in H_{\text{loc}}^s(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$ such that for every chart $\chi : \mathcal{U} \rightarrow B$ on X to $B := \{x \in \mathbb{R}^n : |x| < 1\}$ and every $\varphi \in C_0^\infty(\mathcal{U})$ we have

$$(1 - \omega)\varphi g \in (\beta \circ (1 \times \chi))^* H^s(\mathbb{R}^{n+1})|_\Gamma$$

for $\Gamma := \{(r, rx) \in \mathbb{R}^{n+1} : r \in \mathbb{R}_+, x \in B\}$, $n = \dim X$, and $\beta : \mathbb{R}_+ \times B \rightarrow \Gamma$, $\beta(r, x) := (r, rx)$, $(1 \times \chi)(r, \cdot) := (r, \chi(\cdot))$, see picture below.



The spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ can be endowed with scalar products such that they are Hilbert spaces in a natural way; in particular, $\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(X^\wedge)$.

For a finite weight interval $\Theta = (\vartheta, 0]$ we set

$$(1.2.2) \quad \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) := \varprojlim_{N \in \mathbb{N}} \mathcal{K}^{s,\gamma - \vartheta - \frac{1}{N+1}}(X^\wedge)$$

which is a Fréchet space in the projective limit topology, and

$$\mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge),$$

as a direct sum, for every asymptotic type \mathcal{P} which is associated with the weight data (γ, Θ) . For purposes below for every $N \in \mathbb{N}$ and for $\gamma = 0$, we now form the spaces

$$(1.2.3) \quad B^N := \langle r \rangle^{-N} \mathcal{K}^{N,0}(X^\wedge)$$

and

$$(1.2.4) \quad A_{\mathcal{P}}^N := \langle r \rangle^{-N} \mathcal{K}^{N, -\vartheta - \frac{1}{N+1}}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge) = w^{-\vartheta - \frac{1}{N+1}} B^N + \mathcal{E}_{\mathcal{P}}(X^\wedge);$$

here, $w(r) := 1 + (r-1)\omega(r)$.

These are Hilbert spaces in a natural way, and we set

$$\mathcal{S}^0(X^\wedge) := \varprojlim_{N \in \mathbb{N}} B^N, \quad \mathcal{S}_{\mathcal{P}}^0(X^\wedge) := \varprojlim_{N \in \mathbb{N}} A_{\mathcal{P}}^N.$$

More generally, we can form

$$\mathcal{S}^\gamma(X^\wedge) := w^\gamma \mathcal{S}^0(X^\wedge) \quad \text{and} \quad \mathcal{S}_{T^{-\gamma}\mathcal{P}}^\gamma(X^\wedge) := w^\gamma \mathcal{S}_{\mathcal{P}}^0(X^\wedge),$$

where $T^{-\gamma}\mathcal{P} := \{(p_j - \gamma, m_j, L_j)\}_{j=0, \dots, N}$.

Remark 1.1. (i) *There are canonical continuous embeddings*

$$(1.2.5) \quad A_{\mathcal{P}}^N \hookrightarrow A_{\mathcal{P}}^{N-1}, \quad B^N \hookrightarrow B^{N-1}$$

for all $N \geq 1$;

(ii) *let us set*

$$(1.2.6) \quad (\kappa_\lambda u)(r, x) := \lambda^{\frac{n+1}{2}} u(\lambda r, x),$$

$n = \dim X$, $\lambda \in \mathbb{R}_+$. Then we obtain strongly continuous groups of isomorphisms

$$\kappa_\lambda : A_{\mathcal{P}}^N \rightarrow A_{\mathcal{P}}^N \quad \text{as well as} \quad \kappa_\lambda : B^N \rightarrow B^N$$

for every $N \in \mathbb{N}$ (recall that $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is said to be strongly continuous on a Banach space B if $\lambda \mapsto \kappa_\lambda b$ represents a continuous function $\mathbb{R}_+ \rightarrow B$ for every $b \in B$).

1.3. Green edge symbols. Green symbols will be particular operator-valued symbols within the framework of ‘twisted homogeneity’. Homogeneity in that sense means the following. Let E be a Hilbert space equipped with a strongly continuous group $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : E \rightarrow E$, $\lambda \in \mathbb{R}_+$, such that $\kappa_\lambda \kappa_\rho = \kappa_{\lambda\rho}$ for all $\lambda, \rho \in \mathbb{R}_+$ (in such a case we simply say that E is endowed with a group action). If \tilde{E} is another Hilbert space with group action $\{\tilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}$, a C^∞ function $a_{(\mu)}(y, \eta)$ in $\Omega \times (\mathbb{R}^q \setminus \{0\})$, $\Omega \subseteq \mathbb{R}^q$ open, with values in $\mathcal{L}(E, \tilde{E})$ is called homogeneous of order $\mu \in \mathbb{R}$ if

$$a_{(\mu)}(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y, \eta) \kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$.

Let us give the definition of the space of symbols $S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$. This space consists of the set of all C^∞ functions $a(y, \eta)$ in $\Omega \times \mathbb{R}^q$ with values in $\mathcal{L}(E, \tilde{E})$ such that

$$(1.3.1) \quad \sup_{\substack{y \in K \\ \eta \in \mathbb{R}^q}} \langle \eta \rangle^{|\beta| - \mu} \left\| \tilde{\kappa}_{\langle \eta \rangle}^{-1} \{ D_y^\alpha D_\eta^\beta a(y, \eta) \} \kappa_{\langle \eta \rangle} \right\|_{\mathcal{L}(E, \tilde{E})}$$

is finite for every $K \subset \subset \Omega$ and every $\alpha, \beta \in \mathbb{N}^q$; here $\langle \eta \rangle := (1 + |\eta|^2)^{\frac{1}{2}}$. Moreover, we denote by $S^{-\infty}(\Omega \times \mathbb{R}^q; E, \tilde{E}) := \bigcap_{\mu \in \mathbb{R}} S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$ the space of symbols of order minus infinity.

Symbols of that kind form a Fréchet space with the expressions (1.3.1) as seminorms. They are ‘twisted’ analogues of Hörmander’s symbol spaces from the scalar case (i.e., when $E = \tilde{E} = \mathbb{C}$, $\kappa_\lambda = \tilde{\kappa}_\lambda = \text{id}_{\mathbb{C}}$ for all $\lambda \in \mathbb{R}_+$). Standard manipulations known from the scalar case also make sense in analogous form in the operator-valued case. In particular, we can form asymptotic sums of sequences $a_j(y, \eta)$ of symbols the order of which tend to $-\infty$ as $j \rightarrow \infty$. Now

$$(1.3.2) \quad S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$$

is defined as the subspace of $a(y, \eta) \in S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$ which admit asymptotic expansions into symbols of the kind $\chi(\eta) a_{(\mu-j)}(y, \eta)$, $j \in \mathbb{N}$, where $a_{(\mu-j)}(y, \eta)$ is homogeneous in the above sense, of order $\mu - j$ and $\chi(\eta)$ an excision function, i.e., any $\chi \in C^\infty(\mathbb{R}^q)$ that vanishes near $\eta = 0$ and is equal to 1 for $|\eta| \geq C$ for some $C > 0$.

Remark 1.2. Let $\chi(\eta)$ be an excision function, and let $a_{(\mu)}(y, \eta)$ be homogeneous of order μ as above; then

$$a(y, \eta) := \chi(\eta) a_{(\mu)}(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E}).$$

In symbol spaces of the kind (1.3.2) for E and \tilde{E} we will take, for instance,

$$(1.3.3) \quad \mathcal{K}^{s, \gamma}(X^\wedge) \quad \text{and} \quad \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge),$$

respectively, for some discrete asymptotic type \mathcal{P} (associated to weight data (β, Θ) , cf. Section 1.2). The spaces (1.3.3) will be considered with the group action (1.2.6).

In order to define Green symbols we need a slight generalisation to the case of Fréchet spaces. We say, that a Fréchet space E , written as the projective limit of a sequence of Hilbert spaces E^j , $j \in \mathbb{N}$, with continuous embedding $\dots \hookrightarrow E^{j+1} \hookrightarrow E^j \hookrightarrow \dots \hookrightarrow E^0$ for all j , is endowed with a group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, if $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a group action on E^0 and $\{\kappa_\lambda|_{E^j}\}_{\lambda \in \mathbb{R}_+}$ defines a group action on E^j for every j .

Definition 1.3. An operator function $g(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{0, \gamma}(X^\wedge), \mathcal{K}^{0, \beta}(X^\wedge)))$ is said to be a Green symbol of order $\mu \in \mathbb{R}$, with (discrete) asymptotic types \mathcal{P} and \mathcal{Q} (associated with the weight data (β, Θ) and $(-\gamma, \Theta)$, respectively) if $g(y, \eta)$ has the properties

$$(1.3.4) \quad g(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s, \gamma}(X^\wedge), \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge))$$

and

$$(1.3.5) \quad g^*(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s, -\beta}(X^\wedge), \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge))$$

for all $s \in \mathbb{R}$. Here g^* denotes the (y, η) -wise formal adjoint with respect to the respective sesquilinear pairings

$$\mathcal{K}^{s, \beta}(X^\wedge) \times \mathcal{K}^{-s, -\beta}(X^\wedge) \rightarrow \mathbb{C}$$

induced by the $\mathcal{K}^{0,0}(X^\wedge)$ scalar product, for arbitrary $s, \beta \in \mathbb{R}$.

Remark 1.4. Observe that the Green operators of type 0 in the calculus of classical (pseudo-differential) boundary value problems are operators with special such symbols. In this case it suffices to replace $\mathcal{K}^{s, \gamma}(X^\wedge)$ and $\mathcal{K}^{s, -\beta}(X^\wedge)$ by $L^2(\mathbb{R}_+)$ (with \mathbb{R}_+ being the inner normal to the boundary in consideration) and $\mathcal{S}_P^\beta(X^\wedge)$ and $\mathcal{S}_Q^{-\gamma}(X^\wedge)$ by $\mathcal{S}(\overline{\mathbb{R}}_+)$, cf. [14].

Remark 1.5. The conditions (1.3.4) and (1.3.5) are slightly stronger than necessary. It suffices to require them for $s = 0$; however, this is not the main point of our consideration. What we can see immediately is that it suffices to require the conditions (1.3.4) and (1.3.5) for all $s \in \mathbb{Z}$ owed by the interpolation property of the spaces $\mathcal{K}^{s, \gamma}(X^\wedge)$ in s . It follows that the space of Green symbols of order μ and fixed \mathcal{P}, \mathcal{Q} is a Fréchet space.

From the Green symbols which are known from the calculus of operators on a manifold with edges we know in fact more, namely, that the spaces $\mathcal{K}^{s, \gamma}(X^\wedge)$ and $\mathcal{K}^{s, -\beta}(X^\wedge)$ may even be replaced by $\langle r \rangle^j \mathcal{K}^{s, \gamma}(X^\wedge)$ and $\langle r \rangle^j \mathcal{K}^{s, -\beta}(X^\wedge)$, respectively, for arbitrary $j \in \mathbb{N}$. Therefore, we start with that property. In that case it is known that the kernels of the homogeneous components $g_{(\mu-j)}$ are C^∞ functions of $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ with values in the space

$$\left\{ \mathcal{S}_P^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^{-\gamma}(X^\wedge) \right\} \cap \left\{ \mathcal{S}^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}_Q^{-\gamma}(X^\wedge) \right\}$$

where $\mathcal{S}^\beta(X^\wedge) = \varprojlim_{j \in \mathbb{N}} \langle r \rangle^{-j} \mathcal{K}^{\infty, \beta}(X^\wedge)$ for any $\beta \in \mathbb{R}$. Here $\overline{\mathcal{Q}} := \{(\overline{q}_j, n_j, \overline{L}_j)\}_j$ when

$\mathcal{Q} = \{(q_j, n_j, L_j)\}_j$, and $\widehat{\otimes}_\pi$ denotes the (completed) projective tensor product between the respective Fréchet spaces. In this conclusion we employ the fact that when an operator $g : H \rightarrow F$ is continuous from a Hilbert space H to a nuclear Fréchet space F (written as $\varprojlim_{j \in \mathbb{N}} F_j$ for Hilbert spaces F_j with nuclear embeddings

$F_{j+1} \hookrightarrow F_j$ for all j), the operator g has a kernel in $\varprojlim_{j \in \mathbb{N}} F_j \otimes_H H^* = \varprojlim_{j \in \mathbb{N}} F_j \widehat{\otimes}_\pi H^* = F \widehat{\otimes}_\pi H^*$, cf. [8].

2. DISCRETE ASYMPTOTICS

Green operators of the cone algebra can be described by integral kernels in a specific kind of tensor products between spaces of functions with asymptotics near $r = 0$ and of Schwartz type for $r \rightarrow \infty$. Green symbols of the edge calculus are symbols with values in such operators on the cone, in this section with discrete asymptotics. We derive a characterisation for such symbols in terms of kernels, depending on variables and covariables on the edge, where the edge covariable appear as a product with the axial variables r and r' , respectively.

2.1. Green symbols. Let $f(r, x, r', x'; y, \eta)$ be a function in the space

$$(2.1.1) \quad \left\{ \mathcal{S}_P^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^{-\gamma}(X^\wedge) \right\} \cap \left\{ \mathcal{S}^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}_Q^{-\gamma}(X^\wedge) \right\} \widehat{\otimes}_\pi S_{\text{cl}}^{\mu+n+1}(\Omega \times \mathbb{R}^q),$$

$\gamma, \beta, \mu \in \mathbb{R}$, $n = \dim X$, and let $\eta \mapsto [\eta]$ denote any strictly positive C^∞ function in \mathbb{R}^q such that $[\eta] = |\eta|$ for $|\eta| > C$ for some constant $C > 0$. Form the operator function

$$(2.1.2) \quad g(y, \eta)u(r, x) := \int_X \int_0^\infty f(r[\eta], x, r'[\eta], x'; y, \eta)u(r', x')(r')^n dr' dx'.$$

Then we get a Green symbol of order μ in the sense of Definition 1.3. For purposes below we set

$$(2.1.3) \quad g_f(y, \eta) := g(y, \eta).$$

Theorem 2.1. *Every Green symbol $g(y, \eta)$ of order μ as in Definition 1.3 has a representation of the form (2.1.2) for an element $f(r, x, r', x'; y, \eta)$ in the space (2.1.1).*

Proof. For convenience we consider a Green symbol with constant coefficients, i.e., $g = g(\eta)$ (the straightforward generalisation of arguments to the y -dependent case will be omitted). First observe that a simple composition of g with suitable powers in r and r' , allows us to consider the case $\beta = \gamma = 0$. Moreover, without loss of generality we may assume $\mu = 0$ (it suffices to replace g by $[\eta]^{-\mu}g$). In other words we start with $g \in S_{\text{cl}}^0(\mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge), \mathcal{S}_P^0(X^\wedge))$ with the homogeneous components $g_{(-j)}(\eta)$, $j \in \mathbb{N}$. We use the fact that the series

$$(2.1.4) \quad \tilde{g}_l(\eta) := \sum_{j=l}^\infty \chi\left(\frac{\eta}{c_j}\right)g_{(-j)}(\eta)$$

converges in $S_{\text{cl}}^{-l}(\mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge), \mathcal{S}_P^0(X^\wedge))$ for every $l \in \mathbb{N}$. Here $\chi(\eta)$ is any excision function in \mathbb{R}^q , and c_j are constants tending to ∞ sufficiently fast. Then $g(\eta) - \tilde{g}(\eta)$, for $\tilde{g}(\eta) := \tilde{g}_0(\eta)$, is of order $-\infty$ in the sense of the first part of Definition 1.3. In a similar manner we can proceed with the formal adjoint and choose, if necessary, the constants c_j once again larger, such that $g^*(\eta) - \tilde{g}^*(\eta)$ is of order $-\infty$ in the sense of the second part of Definition 1.3.

Setting

$$(2.1.5) \quad \mathcal{S}_P^0(X^\wedge) \hat{\otimes}_\Gamma \mathcal{S}_Q^0(X^\wedge) := \{\mathcal{S}_P^0(X^\wedge) \hat{\otimes}_\pi \mathcal{S}^0(X^\wedge)\} \cap \{\mathcal{S}^0(X^\wedge) \hat{\otimes}_\pi \mathcal{S}_Q^0(X^\wedge)\},$$

the components $g_{(-j)}(\eta)$ can be identified with an η -dependent kernel function of the form $|\eta|^{n+1-j} e_{(-j)}(r|\eta|, x, r'|\eta|, x'; \frac{\eta}{|\eta|})$, $\eta \neq 0$, for $e_{(-j)}(r|\eta|, x, r'|\eta|, x'; \frac{\eta}{|\eta|}) \in C^\infty(S_\eta^{q-1}, \mathcal{S}_P^0(X^\wedge) \hat{\otimes}_\Gamma \mathcal{S}_Q^0(X^\wedge))$, with S_η^{q-1} being the unit sphere in \mathbb{R}^q , such that

$$(2.1.6) \quad g_{(-j)}(\eta)u(r, x) = |\eta|^{n+1-j} \int_X \int_0^\infty e_{(-j)}(r|\eta|, x, r'|\eta|, x'; \frac{\eta}{|\eta|})u(r', x')(r')^n dr' dx'.$$

If E is a Fréchet space with the countable semi-norm system $(p_k)_{k \in \mathbb{N}}$ we denote by $S^\mu(\mathbb{R}^q, E)$ the set of all $a \in C^\infty(\mathbb{R}^q, E)$ such that

$$\sup_{\eta \in \mathbb{R}^q} \langle \eta \rangle^{-\mu+|\alpha|} p_k(D_\eta^\alpha a) < \infty$$

for all $\alpha \in \mathbb{N}^q$, $k \in \mathbb{N}$. There is then the subspace $S_{\text{cl}}^\mu(\mathbb{R}^q, E)$ of classical E -valued symbols in terms of asymptotic expansions of elements $\chi(\eta)a_{(\mu-j)}(\eta)$ with homogeneous components $a_{(\mu-j)}(\eta) \in C^\infty(\mathbb{R}^q \setminus \{0\}, E)$ of order $\mu - j$.

Setting

$$(2.1.7) \quad h_j(r, x, r', x'; \eta) := \chi\left(\frac{\eta}{c_j}\right)[\eta]^{n+1-j} e_{(-j)}(r, x, r', x'; \frac{\eta}{|\eta|})$$

we obtain elements

$$(2.1.8) \quad h_j \in S_{\text{cl}}^{n+1-j}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)).$$

Choosing the constants $c_j > 0$ increasing sufficiently fast as $j \rightarrow \infty$ we obtain convergence of $a_l(\eta) := \sum_{j=l}^{\infty} h_j(\eta)$ in $S_{\text{cl}}^{n+1-l}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge))$ for every $l \in \mathbb{N}$. Clearly we can take the same constants as in (2.1.4); it suffices to take the maximums of both choices. Note that $h_j(r, x, r', x'; \eta)$ may be replaced by

$$h_j(r[\eta], x, r'[\eta], x'; \eta) = \chi\left(\frac{\eta}{c_j}\right) |\eta|^{n+1-j} e_{(-j)}(r|\eta|, x, r'|\eta|, x'; \frac{\eta}{|\eta|})$$

when we choose c_0 sufficiently large and $c_j > c_0$ for all $j \geq 1$. According to (2.1.3)

we obtain associated Green symbols $g_{h_j}(\eta)$, and $\sum_{j=l}^{\infty} g_{h_j}(\eta)$ converges to $g_{a_l}(\eta)$ in the

Fréchet space of Green symbols of order $-l$ for the given fixed \mathcal{P}, \mathcal{Q} ; this holds for every $l \in \mathbb{N}$. Thus it follows that $c(\eta) := g(\eta) - g_{a_0}(\eta)$ is a Green symbol of order $-\infty$. It remains to prove that there is an $m(r, x, r', x'; \eta) \in \mathcal{S}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge))$ such that $c(\eta) = g_m(\eta)$. The Green symbol $c(\eta)$ is of order $-\infty$; then there is a

$$(2.1.9) \quad k(r, x, r', x'; \eta) \in \mathcal{S}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge))$$

such that

$$c(\eta)u(r, x) = \int_X \int_0^\infty k(r, x, r', x'; \eta) u(r', x') (r')^n dr' dx'.$$

In Lemma 2.2 below we will show that

$$(2.1.10) \quad k\left(\frac{r}{[\eta]}, x, \frac{r'}{[\eta]}, x'; \eta\right) =: m(r, x, r', x'; \eta) \in \mathcal{S}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)).$$

Then we obviously obtain $c(\eta) = g_m(\eta)$. □

Lemma 2.2. *We have (2.1.9) \Rightarrow (2.1.10).*

Proof. The proof is elementary though voluminous. Therefore, we only describe the typical steps. By virtue of (2.1.5) it suffices to show that

$$(2.1.11) \quad k \in \mathcal{S}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge)) \Rightarrow m \in \mathcal{S}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge))$$

and a similar relation for Schwartz functions with values in the second space of (2.1.5). Let us consider, for instance, the case (2.1.11). We now observe that

$$\mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge) = \varprojlim_{N \in \mathbb{N}} A^N \otimes_H B^N$$

for the spaces $A^N := A_{\mathcal{P}}^N$ and B^N , cf. Section 1.2, with \otimes_H being the Hilbert tensor product. Then we have

$$\mathcal{S}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge)) = \varprojlim_{N \in \mathbb{N}} \mathcal{S}(\mathbb{R}^q, A^N \otimes_H B^N).$$

As the semi-norm system for this space we can take

$$(2.1.12) \quad \sup_{\eta \in \mathbb{R}^q} \left\| \langle \eta \rangle^l D_\eta^\beta k \left(\frac{r}{[\eta]}, x, \frac{r'}{[\eta]}, x'; \eta \right) \right\|_{A^N \otimes_H B^N}$$

for all $l, N \in \mathbb{N}$, $\beta \in \mathbb{N}^q$.

It suffices to show that for every l, β, N there are finitely many triples (l', β', N') such that

$$(2.1.13) \quad \sup_{\eta \in \mathbb{R}^q} \left\| \langle \eta \rangle^{l'} D_\eta^{\beta'} k \left(\frac{r}{[\eta]}, x, \frac{r'}{[\eta]}, x'; \eta \right) \right\|_{A^{N'} \otimes_H B^{N'}} < \infty$$

for all those (l', β', N') implies that (2.1.12) is finite.

Let us look at the case $q = 1$ and $n = \dim X = 0$; the general case is completely analogous. For $\beta = 0$ we use the fact that when $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a strongly continuous group of isomorphisms on a Hilbert space E , there are constants $c, M > 0$ such that

$$(2.1.14) \quad \|\kappa_\lambda\|_{\mathcal{L}(E, \tilde{E})} \leq c (\max(\lambda, \lambda^{-1}))^M$$

for all $\lambda \in \mathbb{R}_+$. From (a slight modification of) Remark 1.1 we know that $u(r, x) \mapsto u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$, induces strongly continuous groups of isomorphisms on the spaces A^N and B^N for all $N \in \mathbb{N}$. Then (2.1.14) yields estimates of the kind

$$(2.1.15) \quad \left\| k \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} \leq c \langle \eta \rangle^M \|k(r, r'; \eta)\|_{A^N \otimes_H B^N}$$

for all η , for suitable constants $c, M > 0$, for all $k \in A^N \otimes_H B^N$. This gives us immediately the conclusion (2.1.13) \Rightarrow (2.1.12) with $\beta' = 0$ and $l' = l + M$.

Let us now assume $\beta = 1$. In this case we obtain

$$\frac{d}{d\eta} k \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) = (\{\varphi r \partial_r + \varphi r' \partial_{r'} + \partial_\eta\} k) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right)$$

with a uniformly bounded function $\varphi(\eta)$ and $\partial/\partial\eta$ denoting the derivative in the third variable. Then

$$\begin{aligned} \left\| \langle \eta \rangle^l \left(\frac{d}{d\eta} k \right) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} &\leq c \left\| \langle \eta \rangle^l (r \partial_r k) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} \\ &+ c \left\| \langle \eta \rangle^l (r' \partial_{r'} k) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} + \left\| \langle \eta \rangle^l (\partial_\eta k) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} \end{aligned}$$

with some $c > 0$. The operator $r \partial_r$ is continuous in the sense

$$(2.1.16) \quad r \partial_r : A^N \rightarrow A^{N-1}, \quad B^N \rightarrow B^{N-1}$$

for every $N \geq 1$. In combination with the estimates (2.1.15) this implies

$$\begin{aligned} & \left\| \langle \eta \rangle^l \left(\frac{d}{d\eta} k \right) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} \leq c \left\| \langle \eta \rangle^{l+M} k(r, r'; \eta) \right\|_{A^{N-1} \otimes_H B^N} \\ & + c \left\| \langle \eta \rangle^{l+M} k(r, r'; \eta) \right\|_{A^N \otimes_H B^{N-1}} + \left\| \langle \eta \rangle^{l+M} (\partial_\eta k)(r, r'; \eta) \right\|_{A^N \otimes_H B^N}. \end{aligned}$$

The desired estimate then follows from Remark 1.1 (i). In an analogous manner we can treat the semi-norms with higher η -derivatives. \square

Remark 2.3. Theorem 2.1 remains true in analogous form if we replace $[\eta]$ in the formula (2.1.2) by any other strictly positive C^∞ function $p(\eta)$ such that $c[\eta] \leq p(\eta) \leq c'[\eta]$ for all η , with suitable constants $0 < c < c'$. In particular, we may take $p(\eta) = \langle \eta \rangle$.

2.2. Trace and potential symbols. The Definition 1.3 can be generalised to 2×2 block matrix-valued functions $g(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{0,\gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{K}^{0,\beta}(X^\wedge) \oplus \mathbb{C})$ such that

$$g(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \oplus \mathbb{C})$$

and

$$g^*(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\beta}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge) \oplus \mathbb{C})$$

for all $s \in \mathbb{R}$, with suitable g -dependent discrete asymptotic types \mathcal{P}, \mathcal{Q} (the point-wise adjoints refer to corresponding sesquilinear pairings induced by the scalar product of $\mathcal{K}^{0,0}(X^\wedge) \oplus \mathbb{C}$). In \mathbb{C} we always assume the trivial group action, i.e., κ_λ acts as the identity for all $\lambda \in \mathbb{R}_+$. Writing $g(y, \eta) = (g_{ij}(y, \eta))_{i,j=1,2}$, we call $g_{21}(y, \eta)$ a trace symbol and $g_{12}(y, \eta)$ a potential symbol of order $\mu \in \mathbb{R}$ (of the edge calculus). Clearly $g_{22}(y, \eta)$ is nothing other than a classical scalar symbol of order μ .

Examples of trace and potential symbols may be obtained by functions in

$$(2.2.1) \quad \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge) \widehat{\otimes}_\pi S_{\text{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q) \ni f_{21}(r', x'; y, \eta)$$

and

$$(2.2.2) \quad \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \widehat{\otimes}_\pi S_{\text{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q) \ni f_{12}(r, x; y, \eta),$$

respectively; as usual, $n = \dim X$. The symbols themselves are obtained by integral representations of the kind

$$(2.2.3) \quad g_{21}(y, \eta)u = \int_X \int_0^\infty f_{21}(r'[\eta], x'; y, \eta) u(r', x') (r')^n dr' dx',$$

$u(y, \eta) \in \mathcal{K}^{s,\gamma}(X^\wedge)$, and

$$(2.2.4) \quad g_{12}(y, \eta)c(r, x) = c f_{12}(r[\eta], x; y, \eta),$$

$c \in \mathbb{C}$, respectively.

Theorem 2.4. (i) Every trace symbol $g_{21}(y, \eta)$ can be written in the form (2.2.3) for an element (2.2.1);
(ii) every potential symbol $g_{12}(y, \eta)$ can be written in the form (2.2.4) for an element (2.2.2).

The proof employs analogous arguments as those for Theorem 2.1.

3. CONTINUOUS ASYMPTOTICS

Continuous asymptotics are motivated by variable discrete asymptotics, where in general the asymptotic data depend on the edge variable y (i.e., exponents as well as the occurring logarithmic powers and the coefficients may be discontinuous in y). We give a definition of weighted spaces with continuous asymptotics and write them as projective limits of suitable Hilbert spaces. Moreover we illustrate the connection of such variable asymptotic data with varying and branching poles of meromorphic Mellin symbols of the calculus. Finally we give a kernel characterisation of Green edge symbols with continuous asymptotics similarly as in the discrete case.

3.1. Green symbols. In Section 1.2 we have formulated spaces $\mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)$ with discrete asymptotics of type \mathcal{P} for $r \rightarrow 0$, cf. the formula (1.2.1). As is known, cf. [13], asymptotics of that form can also be written as

$$(3.1.1) \quad u(r, x) \sim \sum_j \langle \zeta_j, r^{-z} \rangle$$

where ζ_j are $C^\infty(X)$ -valued analytic functionals carried by the points $p_j \in \mathbb{C}$ which are of finite order (in fact, derivatives of the Dirac distribution at p_j of order $m_j + 1$ in the notation of the formula (1.2.1)).

For an open $\mathcal{U} \subseteq \mathbb{C}$ and for a Fréchet space E , we denote by $\mathcal{A}(\mathcal{U}, E) = \mathcal{A}(\mathcal{U}) \widehat{\otimes}_\pi E$ the space of all holomorphic E -valued functions in \mathcal{U} . Moreover, let $\mathcal{A}'(K, C^\infty(X)) (= \mathcal{A}'(K) \widehat{\otimes}_\pi C^\infty(X))$ denote the space of all analytic functionals carried by a compact set $K \subset \mathbb{C}$. From generalities on analytic functionals it follows that every $\zeta \in \mathcal{A}'(K, C^\infty(X))$ can be represented in the form

$$(3.1.2) \quad \zeta : h \mapsto \frac{1}{2\pi i} \int_C f(z) h(z) dz$$

for some $f \in \mathcal{A}(\mathbb{C} \setminus K, C^\infty(X))$, where C is a C^∞ curve counter clockwise surrounding K (such that the winding number with respect to every $z \in K$ is equal to 1). In other words, to express (3.1.1) it suffices to represent ζ_j by a meromorphic function with a pole at p_j of order $m_j + 1$ and Laurent coefficients belonging to the space L_j (cf. the notation in Section 1.2).

Now an element $u(r, x) \in \mathcal{K}^{s,\gamma}(X^\wedge)$ is said to have continuous asymptotics (first in a finite weight strip Θ) if there is an element $\zeta \in \mathcal{A}'(K, C^\infty(X))$ for a suitable compact $K \subset \{z : \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$, $n = \dim X$, such that

$$u(r, x) = \omega(r) \langle \zeta, r^{-z} \rangle + u_\Theta(r, x)$$

for some $u_\Theta \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$; here, as usual, ω is a cut-off function.

In order to unify notation in connection with discrete or continuous asymptotics we consider the space

$$(3.1.3) \quad \{\omega(r) \langle \zeta, r^{-z} \rangle : \zeta \in \mathcal{A}'(K, C^\infty(X))\}.$$

The quotient space of (3.1.3) with respect to the equivalence relation $u \sim v \Leftrightarrow u - v \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$ is called a continuous asymptotic type \mathcal{P} , associated with weight data (γ, Θ) . The cut-off function ω is fixed, but the quotient space is independent of ω . Denoting the space (3.1.3) by $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ we then define

$$(3.1.4) \quad \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge)$$

in the Fréchet topology of the non-direct sum. To recall the terminology, the non-direct sum of two Fréchet spaces E and F (embedded in a Hausdorff topological vector space) is defined as $E + F := \{e + f : e \in E, f \in F\}$ endowed with the Fréchet topology from $E + F \simeq E \oplus F / \Delta$, where $\Delta := \{(e, -e) : e \in E \cap F\}$. Note that \mathcal{P} only depends on the set $K \cap \{\operatorname{Re} z > \frac{n+1}{2} - \gamma + \vartheta\}$. For a generalisation to infinite weight intervals we define the system \mathcal{V} of closed subsets $V \subset \mathbb{C}$ such that $V \cap \{c \leq \operatorname{Re} z \leq c'\}$ is compact for every $c \leq c'$. Then for every $V \in \mathcal{V}$ contained in $\{\operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ we can consider $V_\vartheta := V \cap \{\operatorname{Re} z \geq \frac{n+1}{2} - \gamma + \vartheta - 1\}$ and the associated continuous asymptotic type \mathcal{P}_ϑ . We then have continuous embedding

$$\mathcal{K}_{\mathcal{P}_\vartheta}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{K}_{\mathcal{P}_{\vartheta'}}^{s,\gamma}(X^\wedge)$$

for every $\vartheta < \vartheta'$, and we then set $\mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge) := \varprojlim \mathcal{K}_{\mathcal{P}_\vartheta}^{s,\gamma}(X^\wedge)$ in the Fréchet topology of the projective limit taken over any monotonically decreasing sequence of negative reals ϑ . The subscript \mathcal{P} incorporates a continuous asymptotic type associated with (γ, Θ) for $\Theta = (-\infty, 0]$, and stands for the equivalence class represented by such a sequence $\{\mathcal{P}_\vartheta\}$ with ϑ running over any monotone sequence tending to $-\infty$. The equivalence relation just means the equality of the respective projective limits. The set V is called a carrier set of the asymptotic type \mathcal{P} (when $\vartheta = -\infty$, otherwise $V \cap \{\operatorname{Re} z \geq \frac{n+1}{2} - \gamma + \vartheta\}$ is called the carrier of the corresponding \mathcal{P}).

We do not need the sets V in full generality. Let us content ourselves with those V that are convex in imaginary direction, i.e., $z_0, z_1 \in V$ and $\operatorname{Re} z_0 = \operatorname{Re} z_1$ imply $\lambda z_0 + (1-\lambda)z_1 \in V$ for all $0 \leq \lambda \leq 1$. There is then an obvious one-to-one correspondence between such V contained in $\{\operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ and associated continuous asymptotic types by the above construction.

If \mathcal{P} is a continuous asymptotic type, we set

$$\mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge) := \varprojlim_{N \in \mathbb{N}} \langle r \rangle^{-N} \mathcal{K}_{\mathcal{P}}^{\infty,\gamma}(X^\wedge)$$

which is a nuclear Fréchet space in the topology of the projective limit.

Let us make some remarks about the motivation of continuous asymptotics. As noted in the introduction the elliptic regularity of solutions to elliptic equations $Au = f$ on a wedge $X^\wedge \times \Omega$, $\Omega \subseteq \mathbb{R}^q$ open, and A edge-degenerate of the form (0.0.1), contains a statement on asymptotics of $u(r, x, y)$ for $r \rightarrow 0$, even if we are considering C^∞ functions on $X^\wedge \times \Omega$. Similarly as (1.2.1) the asymptotics have the form

$$(3.1.5) \quad u(r, x, y) \sim \sum_j \sum_{k=0}^{m_j(y)} c_{jk}(x, y) r^{-p_j(y)} \log^k r \quad \text{for } r \rightarrow 0,$$

where the exponents $-p_j(y)$ and the numbers $m_j(y)$ are determined by those points $z \in \mathbb{C}$ where the operators (0.0.4) are not bijective, cf. [11]. These points (as well as the $m_j(y)$) may depend on y in a very irregular way. This may happen even for $n = \dim X = 0$. The inverse of (0.0.4) is then a family of meromorphic functions, and the main ingredients of the parametrices P of A are Mellin operators with such symbols. Applying P to functions (say, with compact support with respect to $r \in \mathbb{R}_+$) gives us functions $u(r, x, y)$ of a behaviour like (3.1.5). If we consider the Mellin transform $(M(\omega u))(z, x, y)$ (for any cut-off function $\omega(r)$ on the

half-axis) we obtain a family of meromorphic functions in the complex plane the poles and multiplicities of which inherit the corresponding behaviour of the Mellin symbols. Now the parameter-dependent asymptotics of u can be interpreted in terms of functions $\zeta(y) \in C^\infty(\Omega, \mathcal{A}'(K, C^\infty(X)))$ for suitable compact K , such that $\zeta(y)$ is pointwise discrete and of finite order but of the above mentioned irregular behaviour. Here ‘pointwise discrete’ means that $\langle \zeta(y), r^{-z} \rangle$ has the form (3.1.5) for certain $p_j \in K$, $m_j \in \mathbb{N}$ for every $y \in \Omega$.

Proposition 3.1. *For every continuous asymptotic type \mathcal{P} associated with weight data (γ, Θ) there is a scale of Hilbert spaces $A_{\mathcal{P}}^N$, $N \in \mathbb{N}$, with nuclear embeddings $A_{\mathcal{P}}^N \hookrightarrow A_{\mathcal{P}}^{N-1}$ for every $N \geq 1$ such that*

$$\mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge) = \varprojlim_{N \in \mathbb{N}} A_{\mathcal{P}}^N.$$

The spaces $A_{\mathcal{P}}^N$ can be chosen as continuously embedded subspaces of $\mathcal{K}^{0,\gamma}(X^\wedge)$ such that (1.2.6) induces a strongly continuous group of isomorphisms

$$\kappa_\lambda : A_{\mathcal{P}}^N \rightarrow A_{\mathcal{P}}^N$$

for every $N \in \mathbb{N}$.

Proof. We first assume the weight interval Θ to be finite. Similarly as (3.1.4) we write $\mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge)$ as a (non-direct) sum of Fréchet spaces, namely,

$$\mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge) = \varprojlim_{N \in \mathbb{N}} \langle r \rangle^{-N} \mathcal{K}_{\Theta}^{\infty,\gamma}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge).$$

We then consider the spaces

$$(3.1.6) \quad \tilde{A}_{\mathcal{P}}^N := \langle r \rangle^{-N} \mathcal{K}^{N,\gamma-\vartheta-\frac{1}{N+1}}(X^\wedge) + \mathcal{E}_{\mathcal{P}}^N(X^\wedge).$$

The meaning of the first summand is clear, cf. also the formula (1.2.2); so it remains to define $\mathcal{E}_{\mathcal{P}}^N(X^\wedge)$. Recall that the space (3.1.3) may be described in terms of a compact set $K := V \cap \{\operatorname{Re} z \geq \frac{n+1}{2} - \gamma + \vartheta - 1\}$ for a set $V \subset \{\operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ of the above mentioned kind. Choose any C^∞ curve $C_N \subset \mathbb{C} \setminus K$ counter clockwise surrounding K , such that the winding number with respect to any $z \in K$ is equal to 1 and $\operatorname{dist}(z, K) \leq \frac{1}{N}$ (it is well known that such curves always exist). Then the weighted Mellin transforms $(M_{\gamma-\frac{\vartheta}{2}}u)|_{C_N}$ belong to $C^\infty(C_N)$ (concerning notation about the weighted Mellin transform, cf. Section 1.1). By $\mathcal{E}_{\mathcal{P}}^N(X^\wedge)$ we then denote the completion of $\{(M_{\gamma-\frac{\vartheta}{2}}u)|_{C_N} : u \in \mathcal{E}_{\mathcal{P}}(X^\wedge)\}$ in the norm of $H^N(C_N)$, the Sobolev space of smoothness N on the curve C_N . This is a Hilbert space, and we have $\mathcal{E}_{\mathcal{P}}(X^\wedge) = \varprojlim_{N \in \mathbb{N}} \mathcal{E}_{\mathcal{P}}^N(X^\wedge)$.

It is now clear that the space (3.1.6) is nuclearly embedded into a corresponding space of analogous structure belonging to $N' < N$ when $N - N'$ is sufficiently large. This allows us to find a sequence of N_j , $j \in \mathbb{N}$, with $N_{j+1} > N_j$ such that, if we set $A_{\mathcal{P}}^j := \tilde{A}_{\mathcal{P}}^{N_j}$, we have nuclearity of $A_{\mathcal{P}}^{j+1} \hookrightarrow A_{\mathcal{P}}^j$ for all j . We then have

$$\mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge) = \varprojlim_{j \in \mathbb{N}} A_{\mathcal{P}}^j.$$

It remains to note that the group action $\kappa_\lambda : u(r, \cdot) \mapsto \lambda^{\frac{n+1}{2}} u(\lambda r, \cdot)$ restricts to a group action on the space $A_{\mathcal{P}}^j$ for every j . It suffices to check that for the spaces (3.1.6). The factor $\lambda^{\frac{n+1}{2}}$ is not essential, so we have to look at the influence of

rescaling to the space $\mathcal{E}_{\mathcal{P}}^N(X^\wedge)$. By definition we restrict the Mellin transform to the curve C_N and measure the result in $H^N(C_N)$. The Mellin transform of the rescaled function is obtained by multiplying the original one by λ^{-z} . The continuous dependence of the $H^N(C_N)$ -norm on $\lambda \in \mathbb{R}_+$ is then obvious. For the infinite weight interval $\Theta = (-\infty, 0]$ we first write $\mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge) = \varprojlim_{m \in \mathbb{N}} \mathcal{S}_{\mathcal{P}_{\vartheta_m}}^\gamma(X^\wedge)$ for a sequence of finite $\vartheta_m < 0$ tending to $-\infty$ and form the spaces $A_{\mathcal{P}_{\vartheta_m}}^N$ for every ϑ of this sequence, such that $\mathcal{S}_{\mathcal{P}_{\vartheta_M}}^\gamma = \varprojlim_{N \in \mathbb{N}} A_{\mathcal{P}_{\vartheta_M}}^N$. Then we can set $A_{\mathcal{P}}^N := A_{\mathcal{P}_{\vartheta_N}}^N$. \square

Remark 3.2. Definition 1.3 has an immediate generalisation to Green symbols with continuous asymptotic types \mathcal{P} and \mathcal{Q} , associated with weight data (β, Θ) and $(-\gamma, \Theta)$, respectively.

3.2. Mellin symbols. As noted in the beginning Green operators on a manifold with conical singularities belong to the algebra of cone pseudo-differential operators. Technically they appear as remainders in some typical operations with so called smoothing Mellin operators, cf. (1.1.1), also defined in terms of asymptotic data.

Recall that M denotes the Mellin transform on \mathbb{R}_+ , i.e., $Mu(z) := \int_0^\infty r^{z-1}u(r)dr$ and Mellin operators on an infinite stretched cone X^\wedge occur with operator-valued symbols taking values in the calculus of operators on the cone. In this connection the Mellin symbols depend on edge variables and covariables, and the mapping properties refer to asymptotic data for $r \rightarrow 0$. It is typical that the Mellin amplitude functions are not only defined on $\Gamma_{\frac{1}{2}-\gamma}$ but in the complex z -plane, up to a subset V which encodes asymptotic properties, similarly as in the context of functions with (discrete or continuous) asymptotics. We want to give a definition and then observe the way how Green operators are induced by Mellin operators with asymptotics.

From now on, we assume the sets $V \in \mathcal{V}$ to be convex in imaginary direction. A V -excision function is any $\chi \in C^\infty(\mathbb{C})$ such that $\chi(z) = 0$ when $\text{dist}(z, V) < \varepsilon_0$, $\chi(z) = 1$ for $\text{dist}(z, V) > \varepsilon_1$ for certain $0 < \varepsilon_0 < \varepsilon_1$.

By $M_V^{-\infty}(X)$ we denote the space of all $f(z) \in \mathcal{A}(\mathbb{C} \setminus V, L^{-\infty}(X))$ such that

$$\chi(z)f(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(X))$$

for every V -excision function χ and every real β , uniformly in compact β -intervals. Moreover, let $M_{\mathcal{O}}^\mu(X)$, $\mu \in \mathbb{R}$, denote the space of all $h(z) \in \mathcal{A}(\mathbb{C}, L_{\text{cl}}^\mu(X))$ such that

$$h(z)|_{\Gamma_\beta} \in L_{\text{cl}}^\mu(X; \Gamma_\beta)$$

for every real β , uniformly in compact β -intervals.

The spaces $M_V^{-\infty}(X)$ and $M_{\mathcal{O}}^\mu(X)$ are nuclear Fréchet spaces in a natural way.

Let us set

$$M_V^\mu(X) := M_{\mathcal{O}}^\mu(X) + M_V^{-\infty}(X)$$

in the Fréchet topology of the non-direct sum. Then, for every $f(r, r', z)$ belonging to the space $C^\infty(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}, M_V^\mu(X))$, we can form associated weighted Mellin operators $\text{op}_M^\beta(f)$, for every weight $\beta \in \mathbb{R}$ such that $V \cap \Gamma_{\frac{1}{2}-\beta} = \emptyset$.

Theorem 3.3. *For every $f(r, r', z) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_V^\mu(X))$, $V \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$, $n = \dim X$, the operator $\omega \text{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega}$ (with cut-off functions $\omega, \tilde{\omega}$) induces continuous operators*

$$\omega \text{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega} : \mathcal{K}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{K}^{s-\mu,\gamma}(X^\wedge)$$

and

$$\omega \text{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega} : \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{K}_{\mathcal{Q}}^{s-\mu,\gamma}(X^\wedge)$$

for every $s \in \mathbb{R}$ and every continuous asymptotic type \mathcal{P} with some resulting continuous asymptotic type \mathcal{Q} , associated with the weight data (γ, Θ) for every $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$.

This result is known, cf. [13]. Recall that the main idea of the continuity in spaces with continuous asymptotics is to characterise the Mellin transforms of ωu as holomorphic functions outside the union of V and the carrier set of the asymptotic type \mathcal{P} ; then we obtain another carrier set which just determines the asymptotic type \mathcal{Q} .

Mellin operators as in Theorem 3.3 belong to the ingredients of parametrices of elliptic (pseudo-differential) operators on manifolds with conical singularities (modelled on X^\wedge) or edges (modelled on $X^\wedge \times \Omega$ for some open set $\Omega \subseteq \mathbb{R}^q$). These operators are combined with other operators of the calculus.

To see how Green operators appear in the set-up of conical singularities, consider the following (known) result; for completeness we give a proof here.

Proposition 3.4. *Consider an element $f \in M_V^{-\infty}(X)$; let $j > 0$, $\mu \in \mathbb{R}$, and let $\gamma - j \leq \beta, \delta \leq \gamma$ for some reals β, δ , such that*

$$V \cap \Gamma_{\frac{n+1}{2}-\beta} = V \cap \Gamma_{\frac{n+1}{2}-\delta} = \emptyset.$$

For the operator

$$g := \omega r^{-\mu+j} \text{op}_M^{\beta-\frac{n}{2}}(f)\tilde{\omega} - \omega r^{-\mu+j} \text{op}_M^{\delta-\frac{n}{2}}(f)\tilde{\omega},$$

$n = \dim X$, there are continuous asymptotic types \mathcal{P} and \mathcal{Q} associated with the weight data $(\gamma - \mu, \Theta)$ and $(-\gamma, \Theta)$, respectively, such that

$$(3.2.1) \quad g : \langle r \rangle^l \mathcal{K}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{S}_{\mathcal{P}}^{\gamma-\mu}(X^\wedge),$$

and

$$(3.2.2) \quad g^* : \langle r \rangle^l \mathcal{K}^{s,-\gamma+\mu}(X^\wedge) \rightarrow \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge)$$

are continuous operators for all $s \in \mathbb{R}$, $l \in \mathbb{N}$.

Proof. Let us check the mapping property (3.2.1); the property (3.2.2) can be verified in an analogous manner by passing to formal adjoints of the involved Mellin operators, using the fact that they are of analogous type with resulting ‘adjoint’ Mellin symbols, etc., cf. [13]. By virtue of the fact that the operators contain cut-off functions we immediately see that the factors $\langle r \rangle^l$ are harmless; so we may look at the case $l = 0$. In addition it suffices to assume $\mu = 0$. The operator g is then continuous as a map $\mathcal{K}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{K}^{s,\gamma}(X^\wedge)$ because of the assumed weight conditions. We have

$$(3.2.3) \quad gu(r) = \frac{1}{2\pi i} r^j \int_{\Gamma_{\frac{n+1}{2}-\beta}} r^{-z} f(z) M \tilde{\omega} u(z) dz - \frac{1}{2\pi i} r^j \int_{\Gamma_{\frac{n+1}{2}-\delta}} r^{-z} f(z) M \tilde{\omega} u(z) dz.$$

Let, for instance, $\beta \leq \delta$. Observe that $M\tilde{\omega}u(z)$ for $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$ is holomorphic in $\operatorname{Re} z \geq \frac{n+1}{2} - \gamma$. Thus, because of the position of $\Gamma_{\frac{n+1}{2}-\beta}$ and $\Gamma_{\frac{n+1}{2}-\delta}$ on the right of $\Gamma_{\frac{n+1}{2}-\gamma}$ we can replace the difference of integrals (3.2.3) as an integration over a closed curve C counter clockwise surrounding the compact set $K := V \cap \{\frac{n+1}{2} - \beta < \operatorname{Re} z < \frac{n+1}{2} - \delta\}$. The function $\tilde{f}(z) := f(z)M\tilde{\omega}u(z)$ is holomorphic in the strip outside K . Hence (3.2.3) takes the form (3.1.2) for $h(z) = r^{-z}$, up to the factor $\omega(r)r^j$. We thus obtain altogether $gu(r) = \omega(r)r^j \langle \zeta, r^{-z} \rangle$ for $\zeta \in \mathcal{A}'(K, C^\infty(X))$ which gives us the mapping property (3.2.1), where the asymptotic type \mathcal{P} is represented by the compact set K , cf. the notation in connection with (3.1.3). \square

Let us consider what are called Mellin edge symbols. Such symbols are finite linear combinations of operator families of the form

$$(3.2.4) \quad m(y, \eta) := \omega(r[\eta])r^{-\mu+j} \operatorname{op}_M^{\gamma_j - \frac{\mu}{2}}(f_{j\alpha})(y)\eta^\alpha \tilde{\omega}(r[\eta])$$

for cut-off functions $\omega, \tilde{\omega}$, and $f_{j\alpha}(y) \in C^\infty(\Omega, M_V^{-\infty}(X))$ for a set $V \in \mathcal{V}$ such that $V \cap \Gamma_{\frac{n+1}{2}-\gamma_j} = \emptyset$, $\Omega \subseteq \mathbb{R}^q$ open. In such expressions we have $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^q$, $|\alpha| \leq j$, and the weights $\gamma_j \in \mathbb{R}$ are assumed to satisfy the condition

$$(3.2.5) \quad \gamma - j \leq \gamma_j \leq \gamma$$

for every $j \in \mathbb{N}$. Then (3.2.4) is a C^∞ family of continuous operators

$$m(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{S}^{\gamma-\mu}(X^\wedge),$$

cf. Section 1.2. We have, in fact, more, namely

$$m(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^{\gamma-\mu}(X^\wedge))$$

for every $s \in \mathbb{R}$, cf. notation (1.3.2), and

$$m(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge), \mathcal{S}_{\mathcal{Q}}^{\gamma-\mu}(X^\wedge))$$

for every continuous asymptotic type \mathcal{P} with some resulting continuous asymptotic type \mathcal{Q} (associated with the weight data (γ, Θ) and $(\gamma - \mu, \Theta)$, respectively). Moreover, the pointwise formal adjoint $m^*(y, \eta)$ (cf. also Definition 1.3), has a similar structure as (3.2.4), i.e., we have

$$m^*(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}_{(\mathcal{R})}^{s,-\gamma+\mu}, \mathcal{S}_{(\mathcal{S})}^{-\gamma}(X^\wedge))$$

for all $s \in \mathbb{R}$, where the subscripts mean without or with the corresponding continuous asymptotic types.

There are now several essential operations in the edge symbolic calculus which produce Green symbols in the sense of Remark 3.2. More precisely, we obtain Green symbols $g(y, \eta)$ of the kind

$$(3.2.6) \quad g(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \langle r \rangle^l \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}_{\mathcal{P}}^{\gamma-\mu}(X^\wedge)),$$

such that

$$(3.2.7) \quad g^*(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \langle r \rangle^l \mathcal{K}^{s,-\gamma+\mu}(X^\wedge), \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge))$$

for suitable continuous asymptotic types \mathcal{P} and \mathcal{Q} associated with the weight data $(\gamma - \mu, \Theta)$ and $(-\gamma, \Theta)$, respectively, for all $l \in \mathbb{N}$, $s \in \mathbb{R}$.

Remark 3.5. An element $m(y, \eta)$ of the form (3.2.4) is a Green symbol for every $j > -\vartheta$, where $\Theta = (\vartheta, 0]$ is the finite weight strip which plays the role in the continuous analogue of Definition 1.3.

Another point concerns the fact that there may be different choices of γ_j (when $j > 0$, otherwise for $j = 0$ we have $\gamma_0 = \gamma$) such that (3.2.5) holds. Let $\tilde{\gamma}_j$ denote any other choice. Then we have the following result:

Remark 3.6. Let $j > 0$, and let γ_j and $\tilde{\gamma}_j$ denote different weights satisfying (3.2.5). Consider the operators (3.2.4) for both weights, e.g.,

$$\tilde{m}(y, \eta) := \omega(r[\eta])r^{-\mu+j} \text{op}_M^{\tilde{\gamma}_j - \frac{n}{2}}(f_{j\alpha})(y)\eta^\alpha \tilde{\omega}(r[\eta]).$$

Then we have $m(y, \eta) = \tilde{m}(y, \eta)$ modulo a Green symbol with continuous asymptotics, cf. Remark 3.2.

3.3. Integral representations. Let $f(r, x, r', x'; y, \eta)$ be a function in the space

$$(3.3.1) \quad \{S_{\mathcal{P}}^{\gamma-\mu}(X^\wedge) \widehat{\otimes}_\pi S^{-\gamma}(X^\wedge)\} \cap \left\{S^{\gamma-\mu}(X^\wedge) \widehat{\otimes}_\pi S_{\mathcal{Q}}^{-\gamma}(X^\wedge)\right\} \widehat{\otimes}_\pi S_{\text{cl}}^{\mu+n+1}(\Omega \times \mathbb{R}^q),$$

now for continuous asymptotic types \mathcal{P} and \mathcal{Q} . Then the integral representation

$$(3.3.2) \quad g(y, \eta)u(r, x) := \int_X \int_0^\infty f(r[\eta], x, r'[\eta], x'; y, \eta)u(r', x')(r')^n dr' dx',$$

$n = \dim X$, gives us special Green symbols with the properties (3.2.6) and (3.2.7) for all $l \in \mathbb{N}$, $s \in \mathbb{R}$.

Theorem 3.7. Let $g(y, \eta)$ satisfy the conditions (3.2.6) and (3.2.7) for all $l \in \mathbb{N}$, $s \in \mathbb{R}$. Then there is an $f(r, x, r', x'; y, \eta)$ in the space (3.3.1) such that the integral representation (3.3.2) holds.

Proof. The proof employs analogous steps as that of Theorem 2.1; so we only discuss the main ideas. For simplicity we omit again the y -variable and write the Green symbol $g(\eta)$ as an asymptotic sum of the kind (2.1.4), modulo a Green symbol of order $-\infty$. For the homogeneous components we take the integral representation (2.1.6) for all j , then form the functions (2.1.7) and obtain the symbols (2.1.8). This yields the corresponding analogue of $g_{a_0}(\eta)$ which is of the desired integral form, modulo a Green symbol of order $-\infty$, given by a kernel like (2.1.9). It then remains to show the analogue of Lemma 2.2 for the case with continuous asymptotics. The proof of that is a purely technical (but elementary) construction in terms of the scales of spaces A^N and B^N . The spaces B^N are the same as before, while the A^N are constructed in Proposition 3.1. The main new aspect to be employed in the proof is the first of the relations (2.1.16). In the present case we have to look at (3.1.6). The first summand is as in (1.2.4), and it remains to observe that $-r\partial_r$ transforms the space $\mathcal{E}_{\mathcal{P}}^N(X^\wedge)$ to $\mathcal{E}_{\mathcal{P}}^{N-1}(X^\wedge)$, modulo a flat contribution which is absorbed by the first summand in (3.1.6). Applying $-r\partial_r$ to the second factor of $\omega(r)\langle \zeta, r^{-z} \rangle \in \mathcal{E}_{\mathcal{P}}^N(X^\wedge)$, cf. (3.1.3), we obtain $\omega(r)\langle \zeta, zr^{-z} \rangle$; thus we remain in the space $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ and hence, from the continuity of $\mathcal{E}_{\mathcal{P}}(X^\wedge) \rightarrow \mathcal{E}_{\mathcal{P}}(X^\wedge)$, $\omega(r)\langle \zeta, r^{-z} \rangle \mapsto \omega(r)\langle \zeta, zr^{-z} \rangle$ and the definition of $\mathcal{E}_{\mathcal{P}}^N(X^\wedge)$, we immediately obtain the desired relation, i.e., $-r\partial_r : \mathcal{E}_{\mathcal{P}}^N(X^\wedge) \rightarrow \mathcal{E}_{\mathcal{P}}^{N-1}(X^\wedge)$. The other element of the proof are very close to the ones of Lemma 2.2 and will be omitted. \square

Analogously as the discrete case, cf. Section 2.2, we can consider 2×2 block matrix-valued functions $g(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{0, \gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{K}^{0, \beta}(X^\wedge) \oplus \mathbb{C})$ such that

$$(3.3.3) \quad g(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s, \gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \oplus \mathbb{C})$$

and

$$(3.3.4) \quad g^*(y, \eta) \in S_{\text{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s, -\beta}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge) \oplus \mathbb{C})$$

for all $s \in \mathbb{R}$, with suitable g -dependent continuous asymptotic types \mathcal{P}, \mathcal{Q} . Let $g(y, \eta) = (g_{ij}(y, \eta))_{i,j=1,2}$; then we call $g_{21}(y, \eta)$ a trace symbol and $g_{12}(y, \eta)$ a potential symbol of order $\mu \in \mathbb{R}$, while $g_{22}(y, \eta)$ is nothing other than a classical scalar symbol (of order μ).

Let $f_{21} \in \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge) \widehat{\otimes}_\pi S_{\text{cl}}^{\mu + \frac{n+1}{2}}(\Omega \times \mathbb{R}^q)$ and $f_{12} \in \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \widehat{\otimes}_\pi S_{\text{cl}}^{\mu + \frac{n+1}{2}}(\Omega \times \mathbb{R}^q)$, and consider the integral representations

$$(3.3.5) \quad g_{21}(y, \eta)u = \int_X \int_0^\infty f_{21}(r'[\eta], x'; y, \eta)u(r', x')(r')^n dr' dx',$$

$u(r, x) \in \mathcal{K}^{s, \gamma}(X^\wedge)$, and

$$(3.3.6) \quad g_{12}(y, \eta)c(r, x) = cf_{12}(r[\eta], x; y, \eta),$$

$c \in \mathbb{C}$. Then we have (trace and potential) symbols satisfying the mapping properties (3.3.3) and (3.3.4).

Theorem 3.8. (i) Every trace symbol $g_{21}(y, \eta)$ can be written in the form

$$(3.3.5) \text{ for an element } f_{21}(r', x'; y, \eta) \in \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge) \widehat{\otimes}_\pi S_{\text{cl}}^{\mu + \frac{n+1}{2}}(\Omega \times \mathbb{R}^q);$$

(ii) every potential symbol $g_{12}(y, \eta)$ can be written in the form (3.3.6) for an element $f_{12}(r, x; y, \eta) \in \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \widehat{\otimes}_\pi S_{\text{cl}}^{\mu + \frac{n+1}{2}}(\Omega \times \mathbb{R}^q)$.

This can be proved by analogous arguments as for Theorem 3.7. Here, as usual, $n = \dim X$.

Remark 3.9. (i) Green symbols in the sense of block matrices (3.3.3) can be composed within the respective spaces of Green symbols (with discrete or continuous asymptotics) and the homogeneous principal components behave multiplicatively.

(ii) The classes of Green symbols (with discrete or continuous asymptotics) are closed under asymptotic summation when the involved asymptotic types are the same for all the summands.

Let us conclude this section with a few intuitive remarks on the nature of continuous asymptotics which give rise to some ‘unexpected’ examples of Green, trace, or potential operators in that context. First note that when $\omega(r)$ is any fixed cut-off function and $z \in \mathbb{C}$, $\text{Re } z < \frac{n+1}{2} - \gamma$, then we have $\omega(r)r^{-z}c(x) \in \mathcal{K}^{\infty, \gamma}(X^\wedge)$ for any $c \in C^\infty(X)$. Recall that such functions may be interpreted as singular functions of the discrete cone asymptotics for $r \rightarrow 0$ with exponent $-z$. Now if $\zeta(y) \in C^\infty(\Omega, \mathcal{A}'(K, C^\infty(X)))$ is any family of analytic functionals carried by a compact set $K \subset \{\text{Re } z < \frac{n+1}{2} - \gamma\}$, the function

$$(3.3.7) \quad \omega(r)\langle \zeta(y), r^{-z} \rangle$$

may be interpreted as the linear superposition of singular functions for the discrete asymptotics with the (y -dependent) density $\zeta(y)$. Such ‘densities’ may be organised as follows. Choose an arbitrary function $f(y, z) \in C^\infty(\Omega \times (\mathbb{C} \setminus K), C^\infty(X))$, holomorphic in $z \in \mathbb{C} \setminus K$ that extends for every $y \in \Omega$ to a certain ($C^\infty(X)$ -valued) meromorphic function in $z \in \mathbb{C}$ with poles $p_j(y) \in K$, $j = 1, \dots, N(y)$, of multiplicities $m_j(y) + 1$. These poles including multiplicities may be not constant in y . Then, setting

$$\zeta(y) : h(z) \mapsto \frac{1}{2\pi i} \int_C h(z) f(y, z) dz$$

where $C \subset \{\operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ is a compact (say, C^∞) curve counter clockwise surrounding the set K such that C has the winding number 1 with respect to every point of K , we obtain an element $\zeta(y) \in C^\infty(\Omega, \mathcal{A}'(K, C^\infty(X)))$ such that (3.3.7) has discrete asymptotics of the kind (3.1.5) for every fixed y . Now

$$\mathbb{C} \ni c \mapsto c \cdot \omega(r[\eta])[\eta]^{\mu + \frac{n+1}{2}} \langle \zeta(y), (r[\eta])^{-z} \rangle$$

defines a potential symbol of order μ in the frame of continuous asymptotics which just produces functions with pointwise (in y) discrete but branching asymptotics. In a similar manner we can organise trace symbols which reflect such asymptotics as well as more general Green symbols of that kind. Constructions of that kind may also be found in [13].

4. GREEN OPERATORS

We give an idea on how the Green operators of the edge calculus are organised, and we then have a look at a parameter-dependent variant and formulate a result on kernel cut-off which yields holomorphic dependence on parameters, here for the case of constant discrete or continuous asymptotics. This is done on the level of kernel functions constructed in the preceding sections.

4.1. Green operators on a manifold with edges. Let M be a compact manifold with edge Y , locally near any $y \in Y$ modelled on $X^\Delta \times \mathbb{R}^q$, where X is a closed compact C^∞ manifold. Recall that transition functions between (open) stretched wedges $\mathbb{R}_+ \times X \times \mathbb{R}^q \ni (r, x, y)$ are assumed to be C^∞ up to $r = 0$. In addition we choose the global atlas by such singular charts near Y in such a way that the transition functions are constant with respect to r for $0 < r < \varepsilon$ for some $\varepsilon > 0$. By \mathbb{M} we denote the stretched manifold associated with M , see the introduction.

We consider the weighted edge Sobolev space $\mathcal{W}^{s,\gamma}(\mathbb{M})$ that is defined as the subspace of all $u \in H_{\text{loc}}^s(\text{int}\mathbb{M})$ which locally near Y in the coordinates (r, x, y) belong to $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\Delta))$. Here $\mathcal{W}^s(\mathbb{R}^q, E)$ for a Hilbert space E with group action κ_λ is the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\left\{ \int \langle \eta \rangle^{2s} \left\| \kappa_{\langle \eta \rangle}^{-1} \hat{u}(\eta) \right\|_E^2 d\eta \right\}^{\frac{1}{2}},$$

with $\hat{u}(\eta)$ being the Fourier transform of u in \mathbb{R}^q . In a similar manner we define $\mathcal{W}^s(\mathbb{R}^q, E)$ for a Fréchet space E which is the projective limit of Hilbert spaces E^j with group actions, with continuous embeddings $\dots \hookrightarrow E^{j+1} \hookrightarrow E^j \hookrightarrow \dots \hookrightarrow E^0$ for all $j \in \mathbb{N}$, such that the group action on E^j is the restriction of the one on E^0 for every j . This allows us to define subspaces

$$(4.1.1) \quad \mathcal{W}^s(\mathbb{R}^q, \mathcal{S}_\mathcal{P}^\gamma(X^\Delta))$$

of $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ for any (discrete or continuous) asymptotic type \mathcal{P} , using the fact that $\mathcal{S}_\mathcal{P}^\gamma(X^\wedge)$ is a Fréchet space with group action induced by κ_λ on $\mathcal{K}^{s,\gamma}(X^\wedge)$, $s \in \mathbb{R}$. Globally on \mathbb{M} we then define $\mathcal{W}_\mathcal{P}^{s,\gamma}(\mathbb{M})$ to be the subspace of $\mathcal{W}^{s,\gamma}(\mathbb{M})$ locally near the edge described by (4.1.1).

Note that there is a slightly modified global edge calculus, based on the spaces

$$\mathcal{K}^{s,\gamma;g}(X^\wedge) := \langle r \rangle^{-g} \mathcal{K}^{s,\gamma}(X^\wedge)$$

for $g = s - \gamma$ rather than $\mathcal{K}^{s,\gamma}(X^\wedge)$, and the group action

$$\kappa_\lambda^g : u(r, x) \mapsto \lambda^{g+\frac{n+1}{2}} u(\lambda r, x),$$

$\lambda \in \mathbb{R}_+$, instead of (1.2.6). Edge spaces $\mathcal{W}^s(\mathbb{R}^q, E)$ modelled on $E = \mathcal{K}^{s,\gamma;g}(X^\wedge)$ with alternative group actions of that kind have been suggested in [2], in connection with the solvability of hyperbolic equations. In addition, following a remark of [17], the resulting global edge spaces on \mathbb{W} have then particularly natural invariance properties. All considerations here in connection with Green symbols and Green operators easily generalise to the modified spaces. For simplicity we return to the case without g ; then in order to have invariance of the global objects we need to choose an atlas on \mathbb{W} with some specified behaviour of transition maps.

A Green operator G (of the type of an upper left corner) with (discrete or continuous) asymptotics is an operator that is locally near Y in stretched coordinates (r, x, y) of the form

$$\text{Op}_y(g)u(y) = \int_{\mathbb{R}^q} \int_{\mathbb{R}^q} e^{i(y-y')\eta} g(y, \eta) u(y') dy' d\eta,$$

for a Green symbol $g(y, \eta)$ of order μ , modulo a global smoothing operator. The latter category of operators is characterised by the property to define a continuous map $\mathcal{W}^{s,\gamma}(\mathbb{M}) \rightarrow \mathcal{W}_\mathcal{P}^{\infty,\gamma-\mu}(\mathbb{M})$ for some asymptotic type \mathcal{P} and a similar property of the formal adjoint. The global definition of Green operators is justified by the following remark.

Remark 4.1. *With symbols $g(y, \eta) \in S_{(\text{cl})}^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$ in the sense of the notation in Section 1.3 we can form associated operators*

$$\text{Op}_y(g)u(y) = \int_{\mathbb{R}^q} \int_{\Omega} e^{i(y-y')\eta} g(y, \eta) u(y') dy' d\eta,$$

$d\eta := (2\pi)^{-q} d\eta$. In particular, if $g(y, \eta)$ is a Green symbol of the kind (3.2.6), then for every $\varphi(r) \in C_0^\infty(\mathbb{R}_+)$ the operators $\varphi \text{Op}_y(g)$ and $\text{Op}_y(g)\varphi$ are smoothing on $\mathbb{R}_+ \times \mathbb{R}^q \times X$. In particular, we see that the singularities of Green operators are concentrated on the boundary $\{0\} \times \Omega \times X$ as is expected in analogy to a corresponding behaviour of Green operators in classical boundary value problems.

In fact, let us first note that the operators of multiplication \mathcal{M}_φ by φ generate (non-classical) symbols $\mathcal{M}_\varphi \in S^0(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$ for every $\gamma \in \mathbb{R}$. Moreover, the multiplication by $r^{-N}\varphi$ for any $N \in \mathbb{N}$ is of similar behaviour. Then,

$$\varphi g(y, \eta) = r^{-N} \varphi r^N g(y, \eta) = r^{-N} \varphi[\eta]^{-N} (r[\eta])^N g(y, \eta).$$

Since the order of $(r[\eta])^N g(y, \eta)$ is the same as that of $g(y, \eta)$ and the multiplication by $[\eta]^{-N}$ gives rise to an order shift by $-N$ we obtain that $\varphi g(y, \eta)$ is an operator-valued symbol of order $-\infty$, and hence $\varphi \text{Op}_y(g) = \text{Op}_y(\varphi g)$ is smoothing.

Observe that a Green operator G on \mathbb{M} induces continuous operators

$$\mathcal{W}^{s,\gamma}(\mathbb{M}) \rightarrow \mathcal{W}_{\mathcal{P}}^{s-\mu,\gamma-\mu}(\mathbb{M})$$

for every $s \in \mathbb{R}$, where \mathcal{P} is a (discrete or continuous) asymptotic type associated with G . This is a consequence of general continuity on Sobolev spaces.

Remark 4.2. *Green operators on a (stretched) manifold \mathbb{M} with edges form an algebra, and the composition is compatible with the local symbolic structure; in particular, the homogeneous principal symbols (in the sense of twisted homogeneity) behave multiplicatively.*

4.2. Green operators with parameters. The concept of operator-valued symbols as in Section 1.3 has a parameter-dependent analogue, when we replace the covariable $\eta \in \mathbb{R}^q$ by $(\eta, \lambda) \in \mathbb{R}^q \times \mathbb{R}^l$ and require the symbolic estimates with respect to (η, λ) . In particular, we obtain a generalisation of Definition 1.3 to the λ -dependent case, cf. also Remark 3.2.

The construction of the preceding section then gives us parameter-dependent families of Green operators. According to the iterative concept of building up pseudo-differential calculi on manifolds with higher (polyhedral) singularities we may employ such parameter-dependent families as (operator-valued) symbols of a next generation of operators, for instance, on the infinite (stretched) cone $\mathbb{R}_+ \times \mathbb{M}$ with base \mathbb{M} . Constructions in that sense may be found in the paper [5], in particular, a number of kernel cut-off results for such operator functions.

Kernel cut-offs can be organised on the level of symbols. In order to illustrate the effects we want to consider Green symbols with discrete asymptotics as in Definition 1.3 which belong to spaces of the kind

$$(4.2.1) \quad S_{\text{cl}}^{\mu}(\Omega \times \mathbb{R}^q \times \mathbb{R}^l; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{P}}^{\beta}(X^{\wedge}))$$

such that $g^*(y, \eta, \lambda)$ belongs to corresponding analogue of the space in (1.3.5). For $l = 1$ we also write Γ_{δ} instead of \mathbb{R} when λ is involved in the form $z = \delta + i\lambda$ for some $\delta \in \mathbb{R}$. Moreover, let

$$S_{\text{cl}}^{\mu}(\Omega \times \mathbb{R}^q \times \mathbb{C}; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{P}}^{\beta}(X^{\wedge}))$$

denote the space of all $g(y, \eta, z)$ which are holomorphic in $z \in \mathbb{C}$ such that $g(y, \eta, \delta + i\lambda)$ belongs to (4.2.1) (for $l = 1$) for every $\delta \in \mathbb{R}$, uniformly in compact δ -intervals, and where $g^*(y, \eta, z)$ satisfies an analogous condition.

Theorem 4.3. *For every $\delta \in \mathbb{R}$ there is a continuous map*

$$S_{\text{cl}}^{\mu}(\Omega \times \mathbb{R}^q \times \Gamma_{\delta}; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{P}}^{\beta}(X^{\wedge})) \rightarrow S_{\text{cl}}^{\mu}(\Omega \times \mathbb{R}^q \times \mathbb{C}; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{P}}^{\beta}(X^{\wedge})),$$

$g(y, \eta, \delta + i\lambda) \mapsto h(y, \eta, \delta + i\lambda)$, such that

$$g(y, \eta, z) - h(y, \eta, z)|_{\Omega \times \mathbb{R}^q \times \Gamma_{\delta}} \in S^{-\infty}(\Omega \times \mathbb{R}^q \times \Gamma_{\delta}; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{P}}^{\beta}(X^{\wedge}))$$

for every $s \in \mathbb{R}$ (and such that the pointwise formal adjoint have an analogous property).

Theorem 4.3 can be proved by applying a kernel cut-off argument as used in an analogous context in [5]. A similar result holds for Green symbols with continuous asymptotics.

According to Theorem 2.1 the holomorphic symbol $h(y, \eta, z)$ has a family of integral kernels

$$(4.2.2) \quad f_{\operatorname{Re} z}(r[\eta, \lambda], x, r'[\eta, \lambda], x', y, \eta, \lambda),$$

$\lambda = \operatorname{Im} z$, via a representation of the form (2.1.2). This is valid for every fixed $\operatorname{Re} z$; however, the holomorphic dependence of (4.2.2) on z is by no means obvious. In other words, kernel cut-off constructions which produce a holomorphy in a complex covariable are better applied to the symbols in their original definition rather than their integral kernels.

REFERENCES

- [1] M.S. Agranovich and M.I. Vishik, *Elliptic problems with parameter and parabolic problems of general type*, Uspekhi Mat. Nauk **19**, 3 (1964), 53–161.
- [2] R. Airapetyan and I. Witt, *Propagation of smoothness for edge-degenerate wave equations*, Eighth International Conference in Magdeburg, February/March 2000 (Basel) (H. Freistühler and G. Warnecke, eds.), Hyperbolic Problems: Theory, Numerics, Applications, vol. 140, Birkhäuser, 2001, pp. 11–18.
- [3] L. Boutet de Monvel, *Comportement d'un opérateur pseudo-différentiel sur une variété à bord*, J. Anal. Math. **17** (1966), 241–304.
- [4] S. Coriasco and B.-W. Schulze, *Edge problems on configurations with model cones of different dimensions*, Osaka J. Math. **43** (2006), 1–40.
- [5] G. De Donno and B.-W. Schulze, *Meromorphic symbolic structures for boundary value problems on manifolds with edges*, Math. Nachr. **279**, 4 (2006), 368–399.
- [6] Ch. Dorschfeldt, *Algebras of pseudo-differential operators near edge and corner singularities*, Math. Res., vol. 102, Wiley-VCH, Berlin, Weinheim, 1998.
- [7] G.I. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Math. Monographs, vol. 52, Amer. Math. Soc., Providence, Rhode Island, 1980, Transl. of Nauka, Moskva, 1973.
- [8] H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, 1981.
- [9] S. Rempel and B.-W. Schulze, *Parametrices and boundary symbolic calculus for elliptic boundary problems without transmission property*, Math. Nachr. **105** (1982), 45–149.
- [10] S. Rempel and B.-W. Schulze, *Asymptotics for elliptic mixed boundary problems (pseudo-differential and Mellin operators in spaces with conormal singularity)*, Math. Res., vol. 50, Akademie-Verlag, Berlin, 1989.
- [11] B.-W. Schulze, *Regularity with continuous and branching asymptotics for elliptic operators on manifolds with edges*, Integral Equations Operator Theory **11** (1988), 557–602.
- [12] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Symp. “Partial Differential Equations”, Holzhau 1988, Teubner-Texte zur Mathematik, vol. 112, Teubner, Leipzig, 1989, pp. 259–287.
- [13] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.
- [14] B.-W. Schulze, *The variable discrete asymptotics of solutions of singular boundary value problems*, Operator Theory: Advances and Applications, vol. 57, Birkhäuser Verlag, Basel, 1992, pp. 271–279.
- [15] B.-W. Schulze and N.N. Tarkhanov, *Green pseudodifferential operators on a manifold with edges*, Comm. Partial Differential Equations **23**, 1-2 (1998), 171–200.
- [16] J. Seiler, *The cone algebra and a kernel characterization of Green operators*, Advances in Partial Differential Equations (Approaches to Singular Analysis) (J. Gil, D. Grieser, and Lesch M., eds.), Oper. Theory Adv. Appl., Birkhäuser, Basel, 2001, pp. 1–29.
- [17] N. Tarkhanov, *Harmonic integrals on domains with edges*, Preprint 2004/20, Institut für Mathematik, Potsdam, 2004.
- [18] A. Volpato, *Green operators in the calculus on manifolds with edges*, Ph.D. thesis, University of Torino, 2005.